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Analytic Waves

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Physical aspects of wave theory are discussed. Analytic waves (AW) neatly define the amplitude and frequency of real running waves and generalize and justify some points of wave theory. It is shown that the local group delay averaged in frequency defines the velocity of a wave center at each point. An asymptotic solution is developed for running spectra in slowly varying media. Also, Whitham's method is generalized not only for the frequency but also the amplitude of a wave. The theory is applied to quantum mechanics, and the paradox of tunneling is clarified. This paradox is not specifically quantum but occurs and can be explained in a classical area.

1. INTRODUCTION

We study one-dimensional scalar waves in nonuniform dispersive media that obey the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} L\left(x, \frac{\partial}{\partial t}\right) u = 0 \qquad \text{where} \quad L\left(x, \frac{\partial}{\partial t}\right) = \sum_n \alpha_n(x) \frac{\partial^n}{\partial t^n} \tag{1}$$

Here $u(x, t)$ is a real field (wave) at a point x at an instant t, and c is the limit velocity of propagation (light velocity). Our aim is to explore the amplitude and frequency of running waves.

The linear differential operator L as given in (1) defines dispersion properties of a medium. Writing $u(x, t)$ in the form

$$u(x, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k, \omega) e^{-i(\omega t - kx)} d\omega \, dk \tag{2}$$

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and substituting into (1), we find the dispersion relation that defines the wave number k as a function of ω and x:

$$k^2(x, \omega) = -\frac{1}{c^2} L(x, -i\omega) \qquad \text{or} \qquad k(x, \omega) = \pm\frac{1}{c} \sqrt{-L(x, -i\omega)} \quad (3)$$

Here $\pm$ specifies the direction of propagation. If L is independent of x, the medium is uniform and k depends on ω only. If also $L = \partial^2/\partial t^2$ and $k = \pm\omega/c$, no dispersion exists, and a signal propagates without distortion. Then (1) is the classical wave equation.[2]

In turn, the wave number determines the group speed $v(x, \omega)$ and the group delay $\tau(x, \omega)$:

$$v(x, \omega) = \frac{\partial\omega}{\partial k}, \qquad \tau(x, \omega) = \frac{1}{v} = \frac{\partial k}{\partial\omega} \quad (4)$$

In a strict sense, group speed and delay are defined for harmonic waves of constant frequency. However, they are also used generally when replacing the spectral frequency ω by the local (instantaneous) frequency $\omega(x, t)$. In particular, Whitham's method (Whitham, 1974) discussed in Section 4 determines the local frequency as a function of t and x.

A slowly varying amplitude and frequency of a wave are often advantageous, but within a framework of real waves

$$u(x, t) = a(x, t)\cos \phi(x, t) \quad (5)$$

we do not know what they are: The amplitude $a(x, t)$, phase $\phi(x, t)$, and instantaneous frequency $\omega(x, t) = \partial\phi/\partial t$ are ambiguous since (5) is an equation with two unknowns (a and ϕ). For the same reason, the amplitudes and frequencies of real signals $u(t) = a(t) \cos \phi(t)$ are also ambiguous. The *analytic signal* (AS) introduced in Gabor (1946) is a cogent way for defining them unambiguously.

In Section 2 we discuss the AS and introduce *analytic waves* (AW) as the analogous method for running waves. Then we obtain the following results.

In a uniform dispersive medium, a signal is distorted, but the wave center moves with a constant averaged group delay per unit distance and with a constant velocity. On the other hand, the duration of a wave varies with distance due to the phase interaction in a medium. In nonuniform media, the local averaged group delay defines the velocity of a center at each point. Although a center and duration can be defined for real running waves, they

[2] Many wave equations can be reduced to (1). For the beam equation $\partial^4u/\partial x^4 + \alpha^2(x)\, \partial^2u/\partial t^2 = 0$, the dispersion relation takes the form $k^4 = \alpha^2(x)\omega^2$. Solving it for k^2 and using the correspondence $\partial/\partial x \leftrightarrow ik$ and $\partial/\partial t \leftrightarrow -i\omega$, we obtain equation (1) with $L = \pm i\alpha(x)\, \partial/\partial t$.

are the same as for the AW. We also discuss waves in damping media and a paradox of causality (Section 3).

An asymptotic method is developed for running spectra in slowly varying media with dispersion or damping, and the waves can be found numerically with the FFT. Whitham's approximate method was originally suggested with no explicit definition of a local frequency, but the AW was implied. We justify and modify Whitham's method to find not only the frequency but also the amplitude of a wave. We also generalize this method for slowly varying media (Section 4).

In quantum mechanics, the center of a wave packet represents a classical particle. For *positive* potentials, Schrödinger's wave packets are the AS as time functions, and for particles moving in one direction, they are also the AW. In damping media, classical waves and quantum particles show a paradoxical violation of causality. For a particle tunneling through a barrier, the paradox is concerned with a passage time. A number of approaches have recently been suggested for defining this time (Landauer and Martin, 1994; Kotler and Nitzan, 1988). However, the paradox disappears if we associate the wave packet with an ensemble of particles instead of a single particle (Section 5).

2. ANALYTIC SIGNALS AND ANALYTIC WAVES

In this section, we discuss analytic signals and introduce analytic waves as a generalization for real waves. We also introduce running spectra of the waves and derive equation for the spectra equivalent to (1).

2.1. Analytic Signals

The AS is a *complex* signal $w(t) = u(t) + iv(t) = a(t)e^{i\phi(t)}$ formed from a real signal $u(t)$ by adding its Hilbert transform $v(t) = H[u(t)]$ as the imaginary part. Then, if $U(\omega)$ is the spectrum of $u(t)$, the AS is formed by adding up the spectral components at *positive frequencies*:

$$w(t) = \frac{1}{\pi}\int_0^\infty U(\omega)e^{i\omega t}\, d\omega \tag{6}$$

Two important properties of the Hilbert transform should be mentioned:

- *Harmonic correspondence.* If $u(t) = a\cos(\omega t + \Phi)$ is a harmonic signal of constant $a > 0$, $\omega > 0$, and Φ, then $v(t) = a\sin(\omega t + \Phi)$ and $w(t) = ae^{i(\omega t+\Phi)}$. So the AS provides a common amplitude and frequency for all harmonic processes.
- *Bedrosian's theorem* (Bedrosian, 1963). If the product $l(t)\cdot h(t)$ consists of low-frequency, $l(t)$, and high-frequency, $h(t)$, factors of non-

overlapping spectra, then a low-frequency factor can be taken out of the Hilbert transform, $H[l(t) \cdot h(t)] = l(t) \cdot H[h(t)]$.

2.2. Narrowband and Wideband Modulation

Now we address a signal with amplitude and frequency depending on slow time:

$$u(t) = a(\epsilon t)\cos[t + \Phi(\epsilon t)] = m(\epsilon t)\cos t - n(\epsilon t)\sin t, \qquad \epsilon << 1 \qquad (7)$$

Quadrature functions $m = a\cos\Phi$ and $n = a\sin\Phi$ also depend on ϵt. If they are differentiable r times, then, as is well known, their spectra decrease at high frequencies as $(\epsilon/\omega)^{r+1}$. With a small error ϵ^{r+1}, the spectra are nonoverlapping with a carrier frequency $\omega_0 = 1$, and applying Hilbert transform to (7) and using Bedrosian's theorem, we obtain

$$v(t) = m(\epsilon t)H[\cos t] - n(\epsilon t)H[\sin t]$$

$$= m(\epsilon t)\sin t + n(\epsilon t)\cos t$$

$$w(t) = u + iv = (m + in)e^{it} = a(\epsilon t)e^{i[t+\Phi(\epsilon t)]}$$

So, for slow (narrowband) modulation, the AS defines the amplitude and frequency in the same way as for harmonic signals. *A fortiori*, this is true for smooth signals like $u(t) = \cos(t + \mu\sin\epsilon t)$ when $r = \infty$ and the error is exponentially small.

The AS is employed in radio devices invented long before the AS was. Modulators, detectors, mixers, etc., contain filters separating low-frequency modulation from a carrier. Owing to this, Bedrosian's theorem is applicable, and the amplitude and frequency modulated and detected in reality agree with the AS (Vakman and Vainshtein, 1978). In modern communications, modulation is often wideband, up to the carrier frequency, but Bedrosian's theorem remains applicable, and the AS defines the amplitude and frequency for wideband signals, too. In other fields, such as frequency measurements and nonlinear oscillations, the AS is also advantageous (Vakman, 1994a,b). Obviously, at any fixed x, the AS can also be applied to real waves $u(x, t)$.

2.3. Physical Conditions for Amplitude and Frequency

Only AS meets three reasonable physical conditions:

- *Amplitude continuity.* If a small variation $\delta u(t)$ is added to $u(t)$, then the associated amplitude variation $\delta a(t)$ must also be small: $\delta a \to 0$ for $\delta u \to 0$.
- *Phase independence of scaling.* If a signal $u(t)$ is replaced by $cu(t)$ for a real constant $c > 0$, then the phase and frequency must remain the same.

- *Harmonic correspondence.* The constant amplitude and frequency of a simple sinusoid must retain their values.

Any other amplitude or frequency violates at least one condition and results in irrelevant answers. Shown (Vakman, 1996) for time signals $u(t) = a(t)\cos\phi(t)$, the same conditions are reasonable for spatial signals $u(x) = a(x)\cos\phi(x)$.

2.4. Analytic Waves

Therefore, to define the amplitude and frequency of a real wave, we introduce the *complex* wave $w(x, t) = a(x, t)e^{i\phi(x,t)}$, which is the AS as a function of either t or x. This complex wave is referred to as the *analytic wave* (AW). If a real wave is given as in (2), then according to (6), the associated AW is[3]

$$w(x, t) = a(x, t)e^{i\phi(x,t)} = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty U(k, \omega)e^{-i(\omega t - kx)}dk\,d\omega \tag{8}$$

In time and space, the amplitudes and frequencies of the AW have the same properties as the AS. Multidimensional functions of type (8) have been studied in Hahn (1992).

As mentioned, we can also use the AS taken at a fixed x. For a real harmonic wave $u(x, t) = \cos(\omega_0 t - k_0 x)$, the AS and AW are the same: $w(x, t) = e^{-i(\omega_0 t - k_0 x)}$. By virtue of Bedrosian's theorem, they are also the same for band-limited (in ω and k) or slowly varying running waves. Moreover, (8) contains *positive* wave numbers only, and spectral components move in the positive direction. Therefore, the AW is the AS for any wave running in this direction.

Generally, we introduce analytic waves in both directions

$$w_\pm(x, t) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty U_\pm(k, \omega)e^{-i(\omega t \mp kx)}\,dk\,d\omega \tag{9}$$

and arbitrary waves are superpositions of w_+ and w_-.

2.5. Running Spectra

For a given medium, k and ω are dependent variables connected with (3). Therefore, (8) takes the form

[3]The Fourier transform (6) differs from (2) in the sign of ω. This distinction is traditional for waves and signals, and for the form (2), the AS w should be replaced by its complex conjugate $w^* = u - iv$ with a spectrum of negative frequencies. However, the amplitude and frequency of a real signal remain the same, and we ignore this distinction in (8) correlating any *one-sided* spectrum (at positive or negative frequencies) to the AS.

$$w(x, t) = \frac{1}{\pi^2}\int_0^\infty \int_0^\infty U(k, \omega)\delta[k - k(x, \omega)]e^{-i(\omega t - kx)}\, dk\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty W(x, \omega)e^{-i\omega t}\, d\omega \tag{10}$$

where $W(x, \omega)$ is the spectrum of the AW at a point x. Substituting (10) into equation (1) and using (3) again, we come to an equivalent equation for the spectrum:

$$\frac{\partial^2 W}{\partial x^2} + k^2(x, \omega)W = 0 \tag{11}$$

This is an ordinary differential equation since, for a fixed ω, $\partial/\partial x$ can be replaced by d/dx.

For a *uniform* medium with k independent of x, the general solution of (11)

$$W(x, \omega) = C_+(\omega)e^{ixk(\omega)} + C_-(\omega)e^{-ixk(\omega)} \tag{12}$$

contains two spectra running in opposite directions, and $w_\pm$ in (9) are their Fourier transforms. At $x = 0$, the initial spectra $C_\pm(\omega)$ are arbitrary, which defines arbitrary running waves. Also, it is seen from (12) that $e^{\pm ixk(\omega)}$ is the transfer function of a uniform medium for the wave running in one direction. Its Fourier transform is the Green's function of a medium. In Section 4.1, the solution of (11) will be given for nonuniform media.

3. THE WAVE CENTER AND DURATION

To a certain degree, a running wave can be represented by its traveling center and duration at each point. We will show that the center and duration are the same for real waves and their AW and that the velocity of the center generalizes the group velocity in a medium. We also discuss some paradoxical phenomena in damping media.

3.1. Center and Duration of a Signal

First we consider a real signal $u(t)$ with spectrum $U(\omega) = A(\omega)e^{i\psi(\omega)}$ and group delay $\tau(\omega) = d\psi/d\omega$. We introduce the first and second moments of the signal and its duration as follows:

$$\bar{t} = \frac{\int_{-\infty}^{\infty} tu^2(t)\, dt}{\int_{-\infty}^{\infty} u^2(t)\, dt}, \qquad \overline{t^2} = \frac{\int_{-\infty}^{\infty} t^2u^2(t)\, dt}{\int_{-\infty}^{\infty} u^2(t)\, dt}, \qquad T^2 = \overline{(t - \bar{t})^2} = \overline{t^2} - \bar{t}^2 \tag{13}$$

where an overbar denotes averaging in time. Clearly, the first moment $\bar{t}$ is the time position of a center, and T is the effective (quadratic) duration of a signal.

Further, applying Parseval's equality to the Fourier pairs $u(t) \leftrightarrow U(\omega) = Ae^{i\psi}$ and $itu(t) \leftrightarrow U'(\omega) = [A' + i\tau A]e^{i\psi}$, we obtain in the frequency domain:

$$\bar{t} = \frac{\int_{-\infty}^{\infty} \tau(\omega)A(\omega)^2 \, d\omega}{\int_{-\infty}^{\infty} A(\omega)^2 \, d\omega} = \bar{\bar{\tau}}$$

$$\overline{t^2} = \frac{\int_{-\infty}^{\infty} [A'(\omega)^2/A(\omega)^2 + \tau(\omega)^2]A(\omega)^2 \, d\omega}{\int_{-\infty}^{\infty} A(\omega)^2 \, d\omega} = \overline{\overline{\left(\frac{A'}{A}\right)^2}} + \overline{\overline{\tau^2}} \tag{14}$$

$$T^2 = \overline{\overline{\left(\frac{A'}{A}\right)^2}} + \overline{\overline{\tau^2}} - \bar{\bar{\tau}}^2 = \overline{\overline{\left(\frac{A'}{A}\right)^2}} + \overline{\overline{(\tau - \bar{\bar{\tau}})^2}}$$

Here a double overbar denotes averaging in frequency over the amplitude spectrum $A(\omega)$.

So the time position of a center is the averaged group delay, and the duration depends on amplitude and phase variations in the spectrum. Moreover, the amplitude and phase components are summed in quadrature without interaction.

3.2. Relation to the AS

For real signals, $A(\omega)$ and $\tau(\omega)$ are even functions, and we can replace the limits in (14) by 0, ∞. Then we come to the AS (6), and formulas (14) define its center and duration as well. Thus, a real signal and its AS have the same center and duration.[4] This is also true for the AW.

3.3. Pure Dispersion

In uniform media, according to (12), the running spectrum is $W(x, \omega) = C(\omega)e^{ixk(\omega)}$, and generally, the *complex* wave number $k(\omega) = k_r(\omega) + ik_i(\omega)$ defines the dispersion and damping for each frequency. Then we have

[4] According to (6), the spectrum of the AS is discontinuous at $\omega = 0$ if $U(0) \neq 0$. Then the second moment of the AS $\overline{t^2} = \int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt$ diverges even if the second moment (13) converges. We assume $U(0) = 0$ [since a constant displacement $U(0) \neq 0$ is indistinctive for wave processes], and then the second moment exists, and the center and duration of $w(t)$ are the same as those of $u(t)$ (Kay and Silverman, 1957).

$$A(x, \omega) = A_0(\omega)e^{-xk_i(\omega)}, \qquad \psi(x, \omega) = \psi_0(\omega) + xk_r(\omega) \tag{15}$$

$$\tau(x, \omega) = \frac{\partial\psi}{\partial\omega} = \frac{d\psi_0}{d\omega} + x\frac{dk_r}{d\omega} = \tau_0(\omega) + x\tau_k(\omega) \tag{16}$$

where $A_0(\omega)$ and $\psi_0(\omega)$ are the amplitudes and phases of the initial spectrum $C(\omega)$. With distance, they transform into $A(x, \omega)$ and $\psi(x, \omega)$ according to (15). Also, $\tau_0(\omega)$ is the group delay in the initial spectrum, whereas $\tau_k(\omega)$ is that in a medium per unit distance (dimensions of τ_0 and τ_k are sec and sec/m, respectively).

For a pure dispersive medium of $k_i(\omega) = 0$, amplitudes are conserved, $A(x, \omega) = A_0(\omega)$. Then, from (16) and (14), we obtain the position of the center at a point x:

$$\bar{t}(x) = \overline{\overline{\tau_0}} + x\overline{\overline{\tau_k}} \tag{17}$$

where $\overline{\overline{\tau_0}}$ and $\overline{\overline{\tau_k}}$ *averaged over the initial spectrum* $A_0(\omega)$ are independent of x. Therefore, in a uniform dispersive medium, the center moves with a constant averaged group delay per unit distance. It is well known that *narrowband* signals approximately move with the group speed for the carrier frequency. Relation (17) generalizes that for any signals and distances.

Duration varies with distance. Using (16) and (14), we also obtain

$$T^2(x) = T_0^2 - \frac{\rho^2}{\Delta T_k^2} + \left(x\Delta T_k + \frac{\rho}{\Delta T_k}\right)^2 \tag{18}$$

where

$$T_0^2 = \overline{\overline{\left(\frac{A_0'}{A_0}\right)^2}} + \overline{\overline{(\tau_0 - \overline{\overline{\tau_0}})^2}}, \qquad \Delta T_k^2 = \overline{\overline{(\tau_k - \overline{\overline{\tau_k}})^2}}, \qquad \rho = \overline{\overline{\tau_0\tau_k}} - \overline{\overline{\tau_0}}\cdot\overline{\overline{\tau_k}}$$

Here T_0 is the effective duration of the initial signal, ΔT_k is the variation of the duration due to dispersion (per unit distance), and ρ is a correlation factor between the initial group delay and that in the medium. Everywhere, averaging is done over the initial spectrum.

The correlation factor may be of either sign. If $\rho < 0$, dispersion compensates for the phases of the initial spectrum. Then the signal shortens, and its duration achieves a minimum at $x_0 = -\rho/\Delta T_k^2$. For $x > x_0$, phases are overcompensated, and duration grows again. Negative ρ corresponds to delay opposite to frequency modulation that is typical for time compression (Section 4.6).

3.4. Nonuniform Media

According to zeroth-order solution (24) given below, in pure dispersive *nonuniform* media, amplitudes are also conserved, whereas the phase and group delay are given by

$$\psi(x, \omega) = \psi_0(\omega) + \int_0^x k(x, \omega)\, dx$$

$$\tau(x, \omega) = \tau_0(\omega) + \int_0^x \tau_k(x, \omega)\, dx$$

Then, averaging τ according to (14), we obtain instead of (17)

$$\bar{t}(x) = \bar{\bar{\tau}}_0 + \int_0^x \bar{\bar{\tau}}_k(x)\, dx \tag{19}$$

$$\text{whence} \quad d\bar{t}/dx = \bar{\bar{\tau}}_k(x) \quad \text{and} \quad v(x) = dx/d\bar{t} = 1/\bar{\bar{\tau}}_k(x)$$

Thus, the local group delay (for a fixed x) *averaged in frequency* defines velocity of the center. In optics this is the velocity of a light pulse depending on the refractive index at each point. In quantum mechanics, this is the velocity of a classical particle (Section 5.2).

3.5. Damping and Causality

In damping media, the amplitudes $A(x, \omega)$ vary. Then the averaged group delay depends on x, and motion may be paradoxical.

Let the initial real signal be time-limited so that $u(0, t) = 0$ for $t < 0$. Then, due to relativistic causality, $u(x, t) = 0$ for $t < x/c$, and the first equation (13) gives

$$\bar{t}(x) = \frac{\int_{x/c}^{\infty} t u^2(x, t)\, dt}{\int_{x/c}^{\infty} u^2(x, t)\, dt} \geq \frac{x}{c}, \qquad \text{so that} \quad \frac{x}{\bar{t}(x)} \leq c$$

Thus, causality limits the *mean velocity* $x/\bar{t}$ but not the *instantaneous velocity* $dx/d\bar{t}$. Under causality conditions, the instantaneous velocity of a center may exceed c or be opposite to the mean velocity.

We now calculate the initial velocity of a center. Using (15), we differentiate (14) and (17) with respect to x and, for $x = 0$, obtain

$$\frac{1}{v(0)} = \frac{d\bar{t}(0)}{dx} = \bar{\bar{\tau}}_k - 2(\overline{\overline{\tau_0 k_i}} - \bar{\bar{\tau}}_0 \cdot \bar{\bar{k}}_i) \tag{20}$$

Depending on a correlation factor between τ_0 and k_i, the velocity $v(0)$ may be infinite or negative, but this paradoxical behavior is explainable. Let a wideband chirp signal be applied to a narrowband medium, and let only its early or final part come through. Then the signal shortens, and its center is shifted in time toward the beginning or end. Such a "motion" results not from propagation, but from the suppression of part of the signal. Clearly, if the end or beginning of the signal is cut off, the center shifts in time backward or forward, respectively. In a damping medium, this effect is specific for frequency modulation, where each frequency is related to a certain time. The same paradox arises in quantum mechanics for tunneling through a barrier (Section 5).

3.6. Pure Damping

In the case of pure damping, the wave number is imaginary, so that $k = ik_i(\omega)$, $k_r(\omega) = 0$, and $\tau_k(\omega) = 0$. Therefore, the signal is weakened and distorted due to selective damping but not delayed, and "propagation" takes no time.[5] Possibly we should not interpret such processes as waves. More generally, the wave number may be imaginary for part of the spectrum and real for another part. Then the signal is partly weakened and partly delayed. This leads to considerable distortions that can be misunderstood as a loss of causality. The paradox of tunneling is of this kind.

4. THE WAVE SHAPE

We have considered the center and duration of a running wave, but not its shape. Now we develop two approximate methods for the shape. First an asymptotic solution will be constructed for a spectrum running in a slowly varying medium. Next we generalize Whitham's method for defining not only the frequency, but also the amplitude of a wave.

4.1. Asymptotic Solution

We return to equation (11). For real k^2 (this case includes dispersion and damping, since k may be real or imaginary), we multiply (11) by W^* and, taking the imaginary part, obtain

$$\mathrm{Im}\left(\frac{\partial^2 W}{\partial x^2} W^*\right) = \frac{\partial}{\partial x} \mathrm{Im}\left(\frac{\partial W}{\partial x} W^*\right) = 0$$

Then, writing the spectrum as $W(x, \omega) = A(x, \omega)e^{i\psi(x,\omega)}$, we denote $\chi = \partial\psi/\partial x$ and have

[5]Nonrelativistic conditions are assumed. Otherwise, as mentioned, the finite light speed results in a small delay.

$$\frac{\partial}{\partial x}\,\mathrm{Im}\left(\frac{\partial A}{\partial x}A + i\chi A^2\right) = \frac{\partial}{\partial x}(\chi A^2) = 0$$

whence $\chi A^2 = f(\omega)$ is *independent of* x. So, the running spectrum that satisfied equation (11) takes the form

$$W(x, \omega) = Ae^{i\psi} = \sqrt{\frac{f(\omega)}{\chi(x, \omega)}}\,e^{i\psi}$$

$$= C(\omega)\sqrt{\frac{\chi(0, \omega)}{\chi(x, \omega)}}\exp\left[i\int_0^x \chi(x, \omega)\,dx\right] \tag{21}$$

where $f(\omega)$ is found from the initial spectrum $C(\omega)$ at $x = 0$.

The function χ is still unknown, but substituting (21) into (11), we come to the equation

$$\chi^2 + \frac{1}{2}\frac{\chi''}{\chi} - \frac{3}{4}\left[\frac{\chi'}{\chi}\right]^2 = k^2 \tag{22}$$

where a prime denotes derivation with respect to x. We now suppose that the wave number is slowly varying, $k = k(\epsilon x, \omega)$ with $\epsilon << 1$. Then χ also depends on ϵx, and the terms with derivatives in (22) are of the order ϵ^2. Neglecting them in the first order, we have $\chi = \pm k$, and the second-order solution results by iteration. Finally, we obtain

$$\chi = \begin{cases} \pm k & \text{for the first order} \\ \pm\left[k - \dfrac{1}{4}\dfrac{k''}{k} + \dfrac{3}{8}\left(\dfrac{k'}{k}\right)^2\right] & \text{for the second order} \end{cases} \tag{23}$$

Continuing iterations, one can find higher corrections to χ of the order ϵ^4, ϵ^6, etc.

In contrast to (12), not only the phases, but also the amplitudes of the spectrum vary in nonuniform dispersive media. Equations (21) and (23) define the spectrum, and the AW $w(x, t)$ is available by Fourier transform (10). Clearly, the FFT is an effective numerical method for the waves.

Equation (21) for the spectrum looks like the WKB approximation, but it has another meaning, and the approximation relates to the function χ only. We have found χ asymptotically for slowly varying media. Besides, for $\chi = k$, the singularity at $k(x, \omega) = 0$ is integrable in (10), and the wave is obtainable at turning points. As is well known, the WKB approximation fails at these points. It can also be shown that, in the second order, the singularity is eliminated, and $\chi \neq 0$ even if $k = 0$.

4.2. Zeroth-Order Solution

For $\chi = k$, from (21) we have $\partial W/\partial x = (-k'/2k + ik)W$, and the zeroth-order equation results if we neglect $k' \sim \epsilon$ compared with k^2. Then we obtain instead of (21)

$$\frac{\partial W}{\partial x} = ik(x, \omega)W, \qquad W(x, \omega) = C(\omega) \exp\left[i \int_0^x k(x, \omega)\, dx \right] \tag{24}$$

So, in zeroth order, the amplitudes are conserved even for nonuniform dispersive media. Useful relations can be found within the framework of this approximation, and its accuracy is often acceptable (Sections 3.4, 4.7, 5.3, and 5.5).

4.3. Whitham's Method

This method gives another opportunity, and its basic idea is easy (Whitham, 1974). A real harmonic wave is given by $u(x, t) = \cos(\omega t - kx)$, where ω and k are constants connected with the dispersion relation

$$k = k(\omega) \tag{25}$$

Whitham writes a general real wave in the form (5) and defines its *local frequency* and *local wave number* as

$$\omega(x, t) = \frac{\partial \phi}{\partial t}, \qquad k(x, t) = -\frac{\partial \phi}{\partial x} \tag{26}$$

His crucial assumption is that the *slowly varying* local k and ω are connected with the same dispersion relation (25).[6] That leads to an equation for $\omega(x, t)$ and, finally, determines a traveling wave. From (26) we have

$$\frac{\partial^2 \phi}{\partial x\, \partial t} = \frac{\partial \omega}{\partial x} = -\frac{\partial k}{\partial t}; \qquad \text{therefore,} \quad \frac{\partial \omega}{\partial x} + \frac{\partial k}{\partial t} = 0 \tag{27}$$

On the other hand, from (25) we have $\partial k/\partial t = k'(\omega)\, \partial \omega/\partial t$ and finally obtain

$$\frac{\partial \omega}{\partial x} + \tau(\omega) \frac{\partial \omega}{\partial t} = 0 \tag{28}$$

where $\tau(\omega) = k'(\omega)$ is the group delay (4). The nonlinear equation (28) can easily be solved (see below), which defines the traveling instantaneous frequency.

[6]This assumption implies the AW: the AS provides slow frequency for real signals (Vakman, 1996), and the AW does that for ω and k. We note also that Whitham's equation (28) was originally derived in equivalent form for $k(x, t)$ instead of $\omega(x, t)$.

Obviously, Whitham's method is an approximate quasistationary approach replacing the global spectral frequency ω by the local instantaneous frequency $\omega(x, t)$ [and global wave number k by local $k(x, t)$].

4.4. Modified Method

Two imperfections of the method may be mentioned. First it determines the frequency but not the amplitude of a wave. Next, we do not know what exactly equation (28) describes, since the frequency of a real wave is ambiguous. Using the AW, we specify the frequency and generalize the method for amplitudes.

Writing the AW in the form

$$w(x, t) = a(x, t)e^{i\phi(x,t)} = e^{i\theta(x,t)}$$

we introduce the *complex* phase and *complex* local ω and k as its derivatives,

$$\theta(x, t) = \phi(x, t) - i \ln[a(x, t)], \qquad \omega(x, t) = \frac{\partial\theta}{\partial t}, \qquad k(x, t) = -\frac{\partial\theta}{\partial x} \tag{29}$$

The dispersion relation (25) is valid for complex frequencies [$k(\omega)$ is an analytic function], and replacing ϕ by θ in (27), we see that the same equation (28) is valid for the complex $\omega(x, t)$. Then it determines not only the real frequency, but also the amplitude of the AW (more precisely, its logarithmic derivative).

4.5. Characteristics

Now we point out a method for solving equation (28). The characteristics are the curves on the (x, t) plane for which the function $\omega(x, t)$ given by the equation is constant, and they are defined by

$$\frac{dx}{dt} = \frac{1}{\tau(\omega)} \tag{30}$$

Indeed, from (30) and (28), we have

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial x}\frac{dx}{dt} + \frac{\partial\omega}{\partial t} = \frac{1}{\tau(\omega)}\left[\frac{\partial\omega}{\partial x} + \tau(\omega)\frac{\partial\omega}{\partial t}\right] = 0$$

So, in the characteristic (30), $\omega(x, t)$ is constant. Therefore, $\tau(\omega)$ and dx/dt are also constant, and the characteristic is a straight line given by

$$t = \xi + x\tau[\omega(\xi)] \tag{31}$$

Here ξ is the time point where the characteristic intersects the axis $x = 0$. Since ω is constant in the characteristic, we can take it at ξ and write it as $\omega(\xi)$.

We have solved equation (28). At $x = 0$, the frequency dependence $\omega(\xi)$ is given as the initial condition and, for a complex frequency, it includes the initial amplitude, too. The dispersion relation $\tau(\omega)$ is also given, and equation (31) implicitly determines the dependence $\xi = \xi(x, t)$. Substituting this dependence into $\omega(\xi)$, we obtain the frequency $\omega(x, t)$ for any x and t.

Finally, for the complex $\omega(x, t)$ obtained, we find the complex phase and the AW as follows:

$$\theta(x, t) = \int \omega(x, t)\, dt + g(x), \qquad w(x, t) = e^{i\theta(x,t)} \tag{32}$$

and for pure dispersion, the unknown function $g(x)$ should be found from energy conservation. Then $|w(x, t)|$ defines the amplitude of a running wave, while its frequency results from the complex $\omega(x, t)$ as its real part.

4.6. Time Compression

As illustrated in Fig. 1, each narrow strip of a chirp signal is delayed by $x\tau(\omega)$ according to its frequency. If the group delay $\tau(\omega)$ is opposite to the frequency modulation $\omega(t)$ in the signal, all strips will arrive at some point x at one instant. Then the duration shortens, the amplitude grows, and the signal gets compressed in time at that point. When moving further, each strip is delayed by its time, and the signal broadens again.

This clarifies the idea, but not the amplitude and duration of the wave. However, a modified Whitham method does that easily. For a chirp signal with $\omega(\xi) = \omega_0 + \beta\xi$ and Gaussian amplitude $a(\xi) = e^{-\xi^2/2}$, the complex frequency is $\omega(\xi) = \omega_0 + (\beta + i)\xi$. Then, if the dispersion is linear, $\tau(\omega) = \tau_1 - (\omega - \omega_0)$, equation (31) takes the form

$$t = \xi + x[\tau_1 - (\beta + i)\xi]$$

and gives

$$\xi(x, t) = \frac{t - x\tau_1}{1 - x(\beta + i)}$$

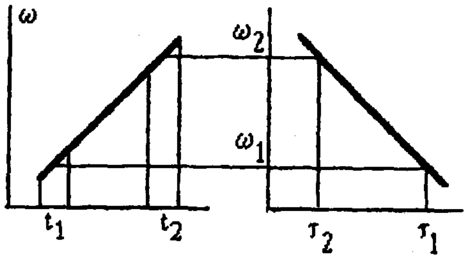

Fig. 1. Compression of a chirp signal.

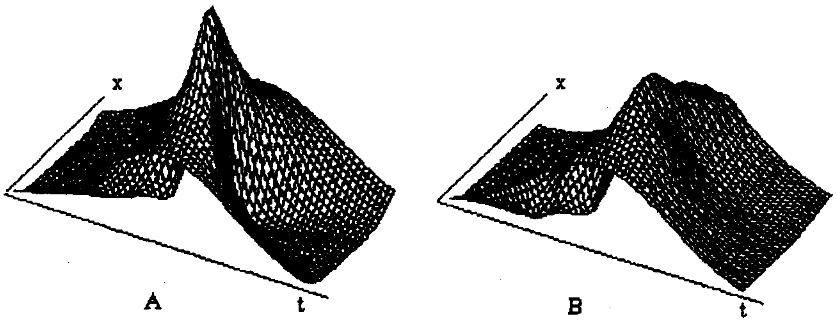

Fig. 2. Amplitudes of chirp signals in a medium with linear dispersion. (A) The chirp signal with a complex frequency $\omega(t) = (5 + i)t$; (B) the same signal with a distorted frequency $\omega(t) = (5 + i)t - 2t^2$.

Therefore, we have

$$\omega(x, t) = \omega_0 + (\beta + i)\frac{t - x\tau_1}{1 - x(\beta + i)}$$

$$\theta(x, t) = \omega_0 t + (\beta + i)\frac{(t - x\tau_1)^2}{2[1 - x(\beta + i)]}$$

Finally, normalizing the AW $w = e^{i\theta}$ to conserve energy at each x, we find the amplitude and duration (see Fig. 2):

$$a(x, t) = \frac{1}{T(x)}\exp\left[-\frac{(t - x\tau_1)^2}{2T^2(x)}\right]$$

$$T^2(x) = \frac{1}{\beta^2 + 1} + (\beta^2 + 1)\left[\frac{\beta}{\beta^2 + 1} - x\right]^2$$

The duration has a minimum at $x = \beta/(\beta^2 + 1)$. The same duration results from (18).

4.7. Whitham's Method for Nonuniform Media

In Whitham's method, derivatives of k and ω are neglected, and for nonuniform media, this method is of zeroth order like (24). For a group delay depending on x, equation (30) takes the form $dx/dt = 1/\tau(x, \omega)$, and the characteristics are not straight lines, but obey the equation

$$t = \xi + \int_0^x \tau[x, \omega(\xi)]\, dx \tag{33}$$

As before, ξ is the point where the characteristic intersects the axis $x = 0$.

Solving (33) for $\xi(x, t)$, one can find the frequency and amplitude of the AW in a nonuniform medium (Section 5.3). The FFT of the spectrum (21) determines the AW more precisely. Practically, numerical solution of (33) is not easier than the FFT, and Whitham's method is mostly expedient in simple cases for analytical solutions.

5. QUANTUM MECHANICAL WAVE PACKETS

Now we address Schrödinger's equation

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + i\hbar\frac{\partial\psi}{\partial t} - V(x)\psi = 0 \tag{34}$$

which defines the quantum mechanical wave function $\psi(x, t)$ for a particle of mass m in a field of potential $V(x)$. The wave center, as defined in Section 3, represents the associated classical particle. *Complex* wave functions $\psi(x, t)$ are often AS and, for particles moving in one direction, they are also AW. That results not from our definition of amplitude and frequency, but from the equation itself. Therefore, possibly quantum mechanics provides a physical basis of AS and AW.

5.1. Dispersion Relation

For equation (34), the general dispersion relation (3) takes the form

$$k(x, \omega) = \pm\frac{\sqrt{2m}}{\hbar}\sqrt{\hbar\omega - V(x)}$$

$$\tau(x, \omega) = \frac{\partial k}{\partial \omega} = \pm\sqrt{\frac{m}{2[\hbar\omega - V(x)]}} \tag{35}$$

Due to the dependence on x of the potential $V(x)$, a wave packet ψ is moving in a nonuniform medium. If $\hbar\omega > V(x)$, k and τ are real, and ψ is propagating in a dispersive medium with the group delay decreasing at high frequencies. If $\hbar\omega < V(x)$, k and τ are imaginary. Then dispersion is replaced by damping, and a wave packet is tunneling through a potential barrier. In both cases, k^2 is real, and the method of Section 4.1 is applicable. Three typical potential functions are shown in Fig. 3. For a free particle when $V(x) = 0$, the medium

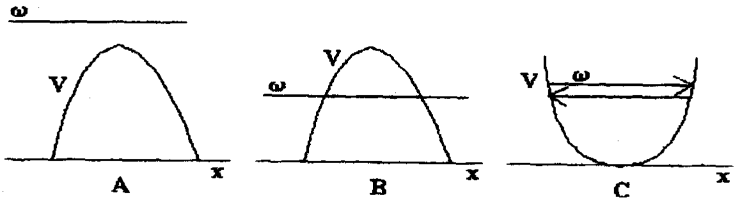

Fig. 3. Potential functions: (A) dispersive motion, (B) tunneling, (C) oscillations.

is uniform, and k and τ are independent of x. Note also that a wideband wave packet may be partly propagating and partly damping.

5.2. Center of a Wave Packet and a Classical Particle

For $\hbar\omega > V$, the center of a wave packet represents the classical particle. In view of (14) and according to (19) and (35), the velocity of the center at a point x is given by

$$v(x) = \frac{1}{\bar{\bar{\tau}}(x)} = \frac{\displaystyle\int A^2(\omega)\, d\omega}{\displaystyle\int \sqrt{m/[2(\hbar\omega - V(x))]}A^2(\omega)\, d\omega} \tag{36}$$

Further, for a *narrowband* spectrum $A(\omega)$ concentrated around ω_0, we replace ω by ω_0 and obtain the classical velocity of a particle as well as energy balance

$$v(x) = \sqrt{\frac{2(\hbar\omega_0 - V(x))}{m}}, \qquad \hbar\omega_0 = \frac{mv^2}{2} + V(x) \tag{37}$$

for the total energy $E = \hbar\omega_0$. So, the velocity of a particle is that of a wave center obtained by averaging over frequency (energy) components.

5.3. Whitham's Approximation

We now apply Whitham's method to equation (34). In view of (35), the equation of characteristics (33) takes the form

$$t = \xi + \sqrt{\frac{m}{2}} \int_0^x \frac{dx}{\sqrt{\hbar\omega(\xi) - V(x)}} \tag{38}$$

This equation has been derived for a wave, but it is the integral of (37) and describes a classical motion. In fact, the classical particle starting from $x = 0$ at an instant ξ with an energy $\hbar\omega(\xi)$ achieves the point x at the instant t given by (38). Nevertheless, this equation also defines the wave function.

If we specify potential in Fig. 3A as

$$V(x) = V_0\left(1 - \frac{x^2}{x_0^2}\right) \qquad \text{for} \quad -x_0 < x < x_0 \tag{39}$$

we find that equation (38) takes the form

$$t = \xi + \sqrt{\frac{mx_0^2}{2V_0}}\left[\operatorname{arsh}\left(\sqrt{\frac{V_0}{\hbar\omega(\xi) - V_0}}\right) + \operatorname{arsh}\left(\frac{x}{x_0}\sqrt{\frac{V_0}{\hbar\omega(\xi) - V_0}}\right)\right]$$

For a Gaussian initial wave packet

$$\psi_0(\xi) = e^{i\omega_0\xi - \xi^2/2T_0^2}$$

at $x = -x_0$, the complex frequency is $\omega(\xi) = \omega_0 + i\xi/T_0^2$, and for $\xi/T_0^2 << \omega_0 - V_0/\hbar$, we obtain an approximate linear equation for ξ. Solving it and normalizing as in Section 4.6, we finally obtain the Gaussian wave function of duration increasing with distance:

$$T^2(x) = T_0^2 + \frac{(x + x_0)^2\hbar m}{8T_0^2\omega_0(\hbar\omega_0 - V_0)^2} \tag{40}$$

For $V_0 = 0$, we also obtain increasing duration for a free particle. It may be mentioned that for a free particle Whitham's method gives an exact solution.

5.4. Interpretation

Why does duration broaden in (40)? From a wave viewpoint, the components of various frequencies are delayed by various times. Therefore, though the initial packet is modulated in amplitude only, frequency modulation appears, which widens the spectrum. However, the spectrum is to be conserved (for the pure dispersion considered), and the duration increases for its narrowing. On the other hand, from a corpuscular viewpoint, particles of various energy (frequency) move with various speeds and disperse over a wide range in time (for a fixed x) or space (for a fixed t).

5.5. Paradox of Tunneling

When a free particle moves toward a barrier, the wave packet broadens, and since the group delay is less at high frequencies, frequency modulation appears with high frequencies at the beginning and low frequencies at the end of the packet. When tunneling, $\hbar\omega < V(x)$, and the wave number is imaginary for the low-frequency part of the spectrum. This part is stopped, whereas the high-frequency part passes above (or through) the barrier.

For the initial chirp packet with Gaussian amplitude and for the potential (39), we have computed the spectra (21) and the wave packets inside the barrier (with the FFT). Due to suppression of low frequencies, the initial spectrum is narrowed (Fig. 4A). High frequencies passing above the barrier reside in the early part of the packet, and therefore the packet shortens while its center is shifted in time toward the beginning. So, the wave center is moving backward (Fig. 4B). Using (13), we have also computed the time positions of the center for the wave packets found in first and zeroth orders [with equation (24) for zeroth order]. Comparing the curves in Fig. 4C, we see that the zeroth-order solution is of acceptable accuracy.

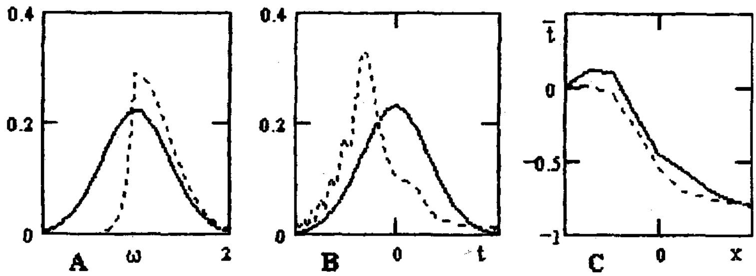

Fig. 4. Tunneling of a chirp wave packet through a barrier. (A, B) Initial (solid line) and final (dashed line) spectra and amplitudes; (C) time positions of the wave center for first (solid line) and zeroth (dashed line) orders. Spectra and wave packets are normalized, and overall damping is ignored.

The center of the transmitted packet leaves the barrier before the center of the incident packet has arrived. That is often understood as a loss of causality. This paradox has given birth to many alternative approaches, and the time of tunneling through a barrier is still an open question (Landauer and Martin, 1994).

We believe that tunneling itself takes no time (see Sections 4.5 and 4.6), and within a corpuscular viewpoint, we may argue in another way. Particles are moving with various speeds, and only faster ones of higher energy have a chance of overcoming the barrier. Therefore, the mean time of arrival of transmitted particles is less than of all incident particles. This mean time is represented by the wave center.

So, the paradox arises if we associate the wave packet with a single particle. However, the wave packet generally represents an ensemble of particles, and the barrier is merely a filter for fast (high-frequency) particles arriving before the others. From a wave viewpoint, this is just the same paradox as in Section 4.5, but no physical conflict emerges.

5.6. Relation to the AS and AW

A rigorous method for defining the spectrum of a wave function is separation of variables, where we seek $\psi(x, t)$ in the form

$$\psi(x, t) = \sum_n c_n T_n(t) X_n(x) \tag{41}$$

and functions $T_n(t)$ and $X_n(x)$ depend on t and x separately. Then it is known (e.g., Saxon, 1968) that Schrödinger's equation (34) is satisfied by the time functions

$$T_n(t) = e^{-i(\lambda_n/\hbar)t} \tag{42}$$

where λ_n are the eigenvalues of Hermitian operator

$$L = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V(x)$$

It is also known that the minimal and maximal values of λ_n are the minimum and maximum of the quadratic form

$$(LX, X) = \int \left[\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} - V(x)X\right]X^* \, dx$$

$$= -\int \left[\frac{\hbar^2}{2m}\left|\frac{dX}{dx}\right|^2 + V(x)|X|^2\right] dx$$

where we have integrated by parts. So, we conclude from the last expression that, for positive potentials $V(x) \geq 0$, the eigenvalues λ_n are *negative*. Then (41) takes the form

$$\psi(x, t) = \sum_n c_n X_n(x) e^{i\omega_n t} \tag{43}$$

and is the AS, since frequencies $\omega_n = -\lambda_n/\hbar$ are positive.

The condition $V(x) \geq 0$ is not met for a Coulomb potential $V(x) = -1/|x|$ and some others, so that wave functions are not AS generally. However, the condition is met for quantum oscillators and other potentials in Fig. 3. In the classical limit, this possibly explains the AS for common oscillators. On the other hand, an oscillating particle corresponds to a standing wave that includes two opposite AW, and spatial frequencies are both positive and negative.

6. SUMMARY

For real signals, amplitude and frequency are ambiguous, and the analytic signal has been introduced as their unambiguous definition. Analytic waves generalize analytic signals for running waves. Using the AW, we have clarified the motion of a wave center and generalized Whitham's method for not only the frequency, but also the amplitude of a wave. The main result obtained is that the group delay *averaged in frequency* defines the velocity of the wave center at each point. We have also developed an asymptotic solution for running spectra in nonuniform media, and the FFT becomes an effective numerical method for the waves.

Quantum mechanical wave packets are often AW or AS, whereas for narrowband packets, the wave center represents the classical particle. In damping media, paradoxical effects arise for classical waves and quantum particles, and seeming violation of causality appears. Interpretation of the paradox is different for waves and particles. In a pure damping medium, the waves are not delayed, and "propagation" takes no time. Also, the conflict disappears if we associate the wave packet with an ensemble of particles instead of a single particle.

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